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New Q Matrices and Their Functional Equations for the Eight Vertex Model at Elliptic Roots of Unity

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Abstract The Q matrix invented by Baxter in 1972 to solve the eight vertex model at roots of unity exists for all values of N, the number of sites in the chain, but only for a subset of roots of unity. We show in this paper that a new Q matrix, which has recently been introduced and is non zero only for N even, exists for all roots of unity. In addition we consider the relations between all of the known Q matrices of the eight vertex model and conjecture functional equations for them.

Keywords TQ equations · Eight vertex models · Functional equations

1 Introduction

In 1972 Baxter [1] invented a method to compute the eigenvalues of the transfer matrix of the eight vertex model without first computing the eigenvectors. This was done by introducing an "auxiliary" matrix $Q(v)$ which satisfies the functional equation

$$T(v)Q(v) = [h(v+\eta)]^N Q(v-2\eta) + [h(v-\eta)]^N Q(v+2\eta) \tag{1}$$

with

$$h(v) = \Theta_m(0)\Theta_m(-v)\mathsf{H}_m(v) \tag{2}$$

where the quasiperiodic theta functions $\Theta_m(v)$, $\mathsf{H}_m(v)$ and the transfer matrix $T(v)$ of the eight vertex model are defined in Appendix 1. The number of lattice sites of the chain with

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periodic boundary conditions is N and $Q(v)$ satisfies the commutation relations

$$[T(v), Q(v')] = 0 \tag{3}$$

$$[Q(v), Q(v')] = 0 \tag{4}$$

Equation (1) is obviously a matrix equation. However, the commutation relations (3) and (4) allow all four matrices in (1) to be simultaneously diagonalized and thus the equation also may be regarded as a scalar functional equation for eigenvalues $t(v)$ and $q(v)$ of the matrices $T(v)$ and $Q(v)$.

For any eigenvalue $t(v)$ the scalar tq equation may be considered to be a second order difference equation for $q(v)$. However, it is important to recognize that in addition to the scalar tq equation quasi-periodicity properties must be independently specified for the functions $q(v)$ in order to obtain explicit solutions and that, as explicitly demonstrated for the eight vertex model in [2], the solutions $q(v)$ to the scalar tq equation do not have to satisfy the same quasi periodicity conditions which are satisfied by the transfer matrix eigenvalues $t(v)$. This difference in quasi-periodicity properties of $t(v)$ and $q(v)$ has recently been studied in [3]. The importance of this is that there are many models such as the SOS [4, 5] and RSOS [6] models for which the eigenvalues of the transfer matrix have been shown to satisfy the scalar tq equation but an operator $Q(v)$ which satisfies a matrix TQ equation is not known. It is therefore most interesting the ask the following question:

What additional information is contained in a Q matrix which is not contained in the scalar tq equation supplemented by the quasiperiodicity properties of the eigenvalues $q(v)$?

This question is particularly relevant to the eight vertex model where the matrices constructed by Baxter in 1972 [1] and in 1973 [7] have been shown [2] to be different. This lack of uniqueness occurs for the eight vertex model when the transfer matrix has degenerate eigenvalues which occur when the parameter η satisfies the "root of unity" condition imposed in the 1972 paper [1]

$$2L_0\eta = 2m_{10}K + im_{20}K' \tag{5}$$

where K (K') are the complete elliptic integrals of the first kind of modulus k (k') and L_0, m_{10} and m_{20} are integers whose greatest common divisor is one. More generally the relation between quasiperiodicity and non-uniqueness of the solutions to the scalar tq equation has been extensively investigated by Bazhanov and Mangazeev [3] for the special case of $m_{20} = 0$.

We have studied the non-uniqueness of Q matrices for the eight vertex model at various roots of unity (5) in a series of papers [2, 8–11] and in [10] and [11] we have seen that there are cases of the root of unity condition (5) where by use of the methods of [1] two different matrices may be constructed, which we call $Q_{72}^{(1)}(v)$ and $Q_{72}^{(2)}(v)$, that are distinct from the matrix $Q_{73}(v)$ constructed by Baxter [7, 14]. One of the distinguishing features is that for different classes of the integers m_{10} and m_{20} the three matrices may have different commutation relations with the three discrete symmetry operators

$$S = \sigma^z \otimes \sigma^z \otimes \cdots \otimes \sigma^z \tag{6}$$

$$R = \sigma^x \otimes \sigma^x \otimes \cdots \otimes \sigma^x \tag{7}$$

and $RS = (-1)^N SR$.

A second most important property of $Q(v)$ matrices which goes beyond the quasiperiodicity properties of the eigenvalues was presented in [2] where it was conjectured that the matrix $Q^{(1)}_{72}(v)$ satisfies a functional equation not involving $T(v)$. This equation is specific to the specific matrix $Q^{(1)}_{72}(v)$ and is NOT a consequence of the scalar tq equation and the quasiperiodicity properties of the eigenvalues of $Q^{(1)}_{72}(v)$. This functional equation is completely analogous to the functional equation first found for the three state chiral Potts model [12]. This analogue between the Q matrix of the eight vertex model and the transfer matrix of the chiral Potts model is presented in great generality in the 1990 paper of Baxter, Bazhanov and Perk [13]. However it is only the matrices $Q^{(1)}_{72}(v)$ and $Q^{(2)}_{72}(v)$ for which this analogy will hold because no such functional equation holds for $Q_{73}(v)$.

The purpose of this present paper is to extend the studies of [2, 8–11] in two ways. The first is to demonstrate that the matrix $Q^{(2)}_{72}(v)$ which was studied in [11] for the case m_{10} and m_{20} both even may be extended to all integer values of m_{10} and m_{20}. The second is to exhibit conjectured functional equations for all cases of the matrices $Q^{(1)}_{72}(v)$ and $Q^{(2)}_{72}(v)$. In Sect. 2 we formulate the problem and summarize the results. The details of the construction of $Q^{(2)}_{72}(v)$ for m_{10} and m_{20} not both even are given in Sect. 3. We conclude in Sect. 4 with a discussion of our results and a few open questions.

2 Formulation and Summary of Results

The construction devised by Baxter in 1972 [1] to find matrices $Q(v)$ which satisfy (1) as summarized in [11] consists of three steps:

(1) The construction of matrices $Q_R(v)$ and $Q_L(v)$ which satisfy

$$T(v)Q_R(v) = \omega^{-N}[h(v+\eta)]^N Q_R(v-2\eta) + \omega^N[h(v-\eta)]^N Q_R(v+2\eta) \tag{8}$$

$$Q_L(v)T(v) = \omega^{-N}[h(v+\eta)]^N Q_L(v-2\eta) + \omega^N[h(v-\eta)]^N Q_L(v+2\eta) \tag{9}$$

where ω is some phase (possibly equal to unity) and $Q_{R,L}(v)$ are of the form

$$Q_{R,L}(v)_{\alpha,\beta} = \mathrm{Tr}[S_{R,L}(\alpha_1,\beta_1)(v)\cdots S_{R,L}(\alpha_N,\beta_N)(v)] \tag{10}$$

where $S_R(\alpha_i,\beta_i)(v)$ are matrices of some dimension $L\times L$ and $\alpha_j,\beta_j=\pm$.

(2) The establishing of interchange relations

$$Q_L(u)\Lambda Q_R(v) = Q_L(v)\Lambda Q_R(u) \tag{11}$$

where there are four values of Λ to be considered

$$\Lambda = I, S, R, RS = (-1)^N SR \tag{12}$$

(3) The construction of $Q_{72}(v)$ from

$$Q_{72}(v) = Q_R(v)Q_R^{-1}(v_0) \tag{13}$$

where v_0 is a value of the spectral parameter v such that $Q_R(v_0)$ is nonsingular. Whenever the interchange relation (11) holds for two different matrices Λ_1 and Λ_2 the matrix $Q_{72}(v)$ will satisfy

$$[Q_{72}(v), \Lambda_1\Lambda_2] = 0 \tag{14}$$

The establishing of these three conditions is sufficient to prove that the $Q(v)$ so defined will satisfy the commutation relations (3) and (4) and the TQ equation (1) with the extra phase ω. If we set

$$Q_{72}(v) = \omega^{-Nv/2\eta}\tilde{Q}_{72}(v) \tag{15}$$

then $\tilde{Q}_{72}(v)$ will satisfy (1) which has no phase factor ω and it is obvious that $\tilde{Q}(v)$ will continue to satisfy the commutation relations (3) and (4).

There are two choices for the matrices $S_{R,L}(\alpha,\beta)(v)$ which have been found to satisfy the requirements of steps 1–3. The first is the choice originally made by Baxter in 1972 [1] where the only non zero elements are $S^{(1)}_{R,L}(\alpha,\beta)_{k,k\pm 1}(v)$, $S^{(1)}_{R,L}(\alpha,\beta)_{0,0}(v)$ and $S^{(1)}_{R,L}(\alpha,\beta)_{L,L}(v)$ and the dimension L is the L_0 of (5). This choice is valid for all N.

The other choice, first found in [10] is valid only for N even (because the trace in (10) vanishes identically for odd N). This choice is given for $k = 1,\dots,L-1$ by

$$S_R{}^{(2)}(+,\beta)_{k,k+1}(v) = -\mathsf{H}_m(v-t-2k\eta)\tau_{\beta,-k} \tag{16}$$

$$S_R{}^{(2)}(+,\beta)_{k+1,k}(v) = \mathsf{H}_m(v+t+2k\eta)\tau_{\beta,\,k} \tag{17}$$

$$S_R{}^{(2)}(-,\beta)_{k,k+1}(v) = \Theta_m(v-t-2k\eta)\tau_{\beta,-k} \tag{18}$$

$$S_R{}^{(2)}(-,\beta)_{k+1,k}(v) = \Theta_m(v+t+2k\eta)\tau_{\beta,\,k} \tag{19}$$

and

$$S_R{}^{(2)}(+,\beta)_{1,L}(v) = \mathsf{H}_m(v+t+2L\eta)\tau_{\beta,\,L} \tag{20}$$

$$S_R{}^{(2)}(+,\beta)_{L,1}(v) = -\mathsf{H}_m(v-t-2L\eta)\tau_{\beta,-L} \tag{21}$$

$$S_R{}^{(2)}(-,\beta)_{1,L}(v) = \Theta_m(v+t+2L\eta)\tau_{\beta,\,L} \tag{22}$$

$$S_R{}^{(2)}(-,\beta)_{L,1}(v) = \Theta_m(v-t-2L\eta)\tau_{\beta,-L} \tag{23}$$

and $S_L^{(2)}$ defined for $k = 1,\dots,L-1$ by

$$S_L{}^{(2)}(\alpha,+)_{k,k+1}(v) = \mathsf{H}_m(v+t+2k\eta)\tau'_{\alpha,-k} \tag{24}$$

$$S_L{}^{(2)}(\alpha,+)_{k+1,k}(v) = -\mathsf{H}_m(v-t-2k\eta)\tau'_{\alpha,\,k} \tag{25}$$

$$S_L{}^{(2)}(\alpha,-)_{k,k+1}(v) = \Theta_m(v+t+2k\eta)\tau'_{\alpha,-k} \tag{26}$$

$$S_L{}^{(2)}(\alpha,-)_{k+1,k}(v) = \Theta_m(v-t-2k\eta)\tau'_{\alpha,\,k} \tag{27}$$

and

$$S_L{}^{(2)}(\alpha,+)_{1,L}(v) = -\mathsf{H}_m(v-t-2L\eta)\tau'_{\alpha,\,L} \tag{28}$$

$$S_L{}^{(2)}(\alpha,+)_{L,1}(v) = \mathsf{H}_m(v+t+2L\eta)\tau'_{\alpha,-L} \tag{29}$$

$$S_L{}^{(2)}(\alpha,-)_{1,L}(v) = \Theta_m(v-t-2L\eta)\tau'_{\alpha,\,L} \tag{30}$$

$$S_L{}^{(2)}(\alpha,-)_{L,1}(v) = \Theta_m(v+t+2L\eta)\tau'_{\alpha,-L} \tag{31}$$

where the dimension L depends on L_0 and the parameters $\tau_{\beta,k}$ and $\tau'_{\alpha,k}$ are arbitrary. The interchange relation (11) will hold only when the parameter t takes on certain specific values.

Table 1 Relation between the parameters occurring in (5) and (37)

	m_{10}	m_{20}	m_1	m_2	L
I	Odd	Even	$2m_{10}$	$2m_{20}$	$2L_0$
II	Odd	Odd	$4m_{10}$	$4m_{20}$	$4L_0$
III	Even	Odd	$4m_{10}$	$4m_{20}$	$4L_0$

We note that

$$Q_L^{(2)}(v;t) = -Q_R^{(2)T}(2K - v;t)S \tag{32}$$

We demonstrated in [11] that there are three subcases of the root of unity condition (5) where $Q_{72}^{(1)}(v)$ satisfies steps 1–3;

$$\text{case I} \quad m_{10} \quad \text{odd} \quad m_{20} \quad \text{even} \tag{33}$$

$$\text{case II} \quad m_{10} \quad \text{odd} \quad m_{20} \quad \text{odd} \tag{34}$$

$$\text{case III} \quad m_{10} \quad \text{even} \quad m_{20} \quad \text{odd} \tag{35}$$

Furthermore in [11] we demonstrated for case of m_{10} and m_{20} both even where $Q_{72}^{(1)}(v)$ does not exist that $Q_{72}^{(2)}(v)$ does satisfy steps 1–3 for the two cases of $t = n\eta$ and $t = (n + 1/2)\eta$ and in addition when $t = (n + 1/2)\eta$ that there are four subcases according to

$$m_{10}, m_{20} \equiv 0, 2 \pmod 4 \tag{36}$$

In this paper we show that the construction of $Q_{72}^{(2)}(v)$ of [11] with $t = n\eta$ may be extended to the cases m_{10} and m_{20} not both even. To achieve this generalization we need to allow the dimension L of the matrices $S_{R,L}^{(2)}(\alpha, \beta)$ to be a multiple of the L_0 defined by (5). Thus we rewrite (5) as

$$2L\eta = 2m_1 K + im_2 K' \tag{37}$$

where now L, m_1 and m_2 are allowed to have common divisors. We find that steps 1–3 are satisfied when L, m_1 and m_2 are given in terms of L_0, m_{10} and m_{20} as shown in Table 1.

As examples we have:

Case I: $\eta = K/3 + iK'/3$, $m_1 = 2, m_2 = 4, L = 6$
Case II: $\eta = K/3 + iK'/6$, $m_1 = 4, m_2 = 4, L = 12$
Case III: $\eta = K + iK'/4$, $m_1 = 8, m_2 = 4, L = 8$

We summarize all of these results in Tables 2–5 where we give the matrices Λ which satisfy the interchange relation (11) and indicate the cases where $Q_R(v)$ is nonsingular.

2.1 Quasiperiodicity of $Q_{72}^{(2)}(v;n\eta)$ for m_{10} and m_{20} Not Both Even

The quasiperiodicity properties of $Q_{72}^{(2)}(v;n\eta)$ are expressed in terms of

$$\omega_1 = 2(r_1 K + ir_2 K'), \qquad \omega_2 = 2(bK + iaK') \tag{38}$$

where r_1 and r_2 are defined by

$$2m_{10} = r_0 r_1, \qquad m_{20} = r_0 r_2 \tag{39}$$

Table 2 The interchange properties and the nonsingularity properties of $Q_R^{(1)}(v)$. We indicate by Y (or N) that the interchange relation with Λ holds (or fails). We indicate by Y (or N) that the inverse of $Q_R^{(1)}(v)$ exists (or fails to exist)

m_{10}	m_{20}	I	S	R	RS	$Q_R^{(1)-1}$
Odd	Even	Y	Y	N	N	Y
Odd	Odd	N	Y	Y	N	Y
Even	Odd	Y	N	Y	N	Y
Even	Even	Y	Y	Y	Y	N

Table 3 The interchange properties of $Q_R^{(2)}(v; n\eta)$ for m_{10} and m_{20} both even. We indicate by Y (or N) whether the interchange relation with Λ holds (or fails) and the notation 0(2) stands for $\equiv 0(2) \pmod 4$. In all cases the matrix $Q_R^{(2)}(v; n\eta)$ is nonsingular

m_{10}	m_{20}	I	S	R	RS
0	0	Y	Y	N	N
2	0	Y	Y	N	N
0	2	Y	Y	N	N
2	2	Y	Y	N	N

Table 4 The interchange properties and nonsingularity properties of $Q_R^{(2)}(v; (n + 1/2)\eta)$ for m_{10} and m_{20} both even. We indicate by Y (or N) that the interchange relation with Λ holds (or fails). We indicate by Y (or N) that the inverse of $Q_R^{(2)}(v; (n+1/2)\eta)$ exists (or fails to exist). The notation 0(2) stands for $\equiv 0(2) \pmod 4$

m_{10}	m_{20}	I	S	R	RS	$Q_R^{(2)-1}$
0	0	Y	Y	N	N	Y
2	0	Y	Y	Y	Y	N
0	2	N	Y	Y	N	Y
2	2	Y	N	Y	N	Y

Table 5 The interchange properties of $Q_R^{(2)}(v; t)$ with m_{10} and m_{20} not both even. We indicate by Y (or N) that the interchange relation with Λ holds (or fails). We indicate by Y (or N) that the inverse of $Q_R^{(2)}(v; t)$ exists (or fails to exist)

m_{10}	m_{20}	L_0	m_1	m_2	L	t	I	S	R	RS	$Q_R^{(2)-1}(v; t)$
o	e	e	$2m_{10}$	$2m_{20}$	$2L_0$	$2n\eta$	Y	Y	N	N	Y
o	e	e	$2m_{10}$	$2m_{20}$	$2L_0$	$(2n+1)\eta$	Y	Y	Y	Y	N
o	e	o	$2m_{10}$	$2m_{20}$	$2L_0$	$2n\eta$	Y	Y	Y	Y	N
o	e	o	$2m_{10}$	$2m_{20}$	$2L_0$	$(2n+1)\eta$	Y	Y	N	N	Y
o	o	e,o	$4m_{10}$	$4m_{20}$	$4L_0$	$n\eta$	Y	Y	Y	Y	Y
e	o	e,o	$4m_{10}$	$4m_{20}$	$4L_0$	$n\eta$	Y	Y	Y	Y	Y

with r_0 the greatest common divisor in $2m_{10}$ and m_{20} and a and b are the integer solutions of

$$ar_1 - br_2 = 1 \tag{40}$$

We note the inverse relations

$$2K = a\omega_1 - r_2\omega_2, \qquad 2iK' = -b\omega_1 + r_1\omega_2 \tag{41}$$

the relation

$$4L_0\eta = r_0\omega_1 \tag{42}$$

and that for $m_{20} = 0$ we have

$$r_0 = 2m_{10}, \qquad r_1 = a = 1, \qquad r_2 = b = 0, \qquad \omega_1 = 2K, \qquad \omega_2 = 2iK' \tag{43}$$

The following results are derived in Appendix 3.

Case I: For m_{10} odd and m_{20} even

$$Q_{72}^{(2)}(v+\omega_1; n\eta) = SQ_{72}^{(2)}(v; n\eta) \tag{44}$$

$$Q_{72}^{(2)}(v+\omega_2; n\eta) = q'^{-N(1+r_2)}e^{-2\pi iNv/\omega_1}S^bQ_{72}^{(2)}(v; n\eta) \tag{45}$$

where

$$q' = e^{i\pi\omega_2/\omega_1} \tag{46}$$

The area of the fundamental region $0, \omega_1, \omega_1+\omega_2, \omega_2$ is $4KK'$.

If we note the definition (15) of $\tilde{Q}^{(2)}(v)$ we see from (44) and (45)

$$\tilde{Q}_{72}^{(2)}(v+\omega_1; n\eta) = S\tilde{Q}_{72}^{(2)}(v; n\eta) \tag{47}$$

$$\tilde{Q}_{72}^{(2)}(v+\omega_2; n\eta) = q'^{-N}e^{-2\pi iNv/\omega_1}S^b\tilde{Q}_{72}^{(2)}(v; n\eta) \tag{48}$$

which are identical with the quasiperiodicity relations of $Q_{72}^{(1)}(v)$ for N even which are reviewed in Appendix 3.

Cases II and III: For m_{20} odd

$$Q^{(2)}(v+\omega_1/2; n\eta) = i^N e^{i\pi Nr_1r_2/4}RS^{r_1/2}Q^{(2)}(v; n\eta) \tag{49}$$

$$Q^{(2)}(v+\omega_2; n\eta) = q'^{-N(1+r_2)}e^{-2\pi iNv/\omega_1}S^bQ^{(2)}(v; n\eta) \tag{50}$$

where we note that it follows from (39) that r_0 and r_2 must be odd and r_1 must be even. The size of the fundamental region is $2KK'$. This is one half of the fundamental region of case I and is the size of the fundamental region of $Q_{73}(v)$.

2.2 Degenerate Eigenvalues of $Q_{72}^{(2)}(v; n\eta)$ for m_{20} Odd

We have investigated the case $\eta = K/2 + iK'/4$ ($m_{10} = m_{20} = 1$) with $N = 12$ and have discovered that $Q_{72}^{(2)}(v; n\eta)$ has 32 pairs of degenerate eigenvalues. The existence of degenerate eigenvalues of a $Q(v)$ matrix is a new phenomenon not previously seen. We take this to be evidence to support the following

Conjecture

(1) The matrices $Q_{72}^{(2)}(v;n\eta)$ for m_{20} odd always have degenerate eigenvalues if N is sufficiently large.

We also note from Tables 2–5 that the only $Q_{72}(v)$ matrix which we have constructed using the procedure of Baxter's 1972 paper [1] that shares the property with the matrix $Q_{73}(v)$ constructed in [7] of commuting with all three discrete symmetry operators S, R and RS is $Q_{72}^{(2)}(v;n\eta)$ with m_{20} odd. However, $Q_{73}(v)$ and $Q_{72}^{(2)}(v;n\eta)$ for m_{20} odd are fundamentally different because $Q_{73}(v)$ has no degenerate eigenvalues.

2.3 Bethe Roots and L Strings

The eigenvalues $q(v)$ of any matrix $Q_{72}(v)$ are quasiperiodic functions which may be characterized by the positions v_j of their zeros and from the scalar tq equation these positions satisfy the equation

$$0 = h^N(v_j+\eta)q(v_j-2\eta) + h^N(v_j-\eta)q(v_j+2\eta) \tag{51}$$

There are two ways in which (51) can be satisfied.

I. If $q(v_j)=0$ and $q(v_j \pm 2\eta) \neq 0$ then we may write (51) as

$$\left(\frac{h(v_j+\eta)}{h(v_j-\eta)}\right)^N = -\frac{q(v_j+2\eta)}{q(v_j-2\eta)} \tag{52}$$

This equation is referred to as "Bethe's equation" and the v_j must lie in the fundamental region of the quasiperiodic function $h^N(v)$. We refer to these roots as Bethe roots and denote them at v_j^B.

II. In addition to these Bethe roots there may be sets containing L roots v_j of the form

$$v_{j;k} = v_{j;0} + 2k\eta \quad 0 \le k \le L-1 \tag{53}$$

for which $q(v_{j;k}) = q(v_{j,k} \pm 2\eta) = 0$ and thus (51) is identically satisfied for any $v_{j;0}$. We refer to these sets of L roots v_j as L strings. The parameters $v_{j;0}$ will lie in the fundamental region of $Q_{72}(v)$. These L strings will cancel out from the scalar tq equation and as a result the eigenvalues $t(v)$ of $T(v)$ are independent of $v_{j;0}$.

Each eigenvalue $q(v)$ of $Q_{72}(v)$ may be factorized as

$$q(v) = q_B(v)q_L(v) \tag{54}$$

where $q_B(v)$ contains the Bethe roots v_j^B which are determined from (52) and $q_L(v)$ contains the L strings whose centers $v_{j;0}$ are not determined from (52). The quasiperiodicity properties of $q_B(v)$ will in general vary from eigenvalue to eigenvalue because the number of Bethe roots will in general be different in each $q_B(v)$. The construction of [4, 5, 15] of a matrix whose eigenvalues are $q_B(v)$ relies on the explicit construction of the eigenvectors of $T(v)$ whose eigenvalue $t(v)$ satisfies the scalar tq equation with $q_B(v)$. This is the opposite of the construction of either $Q_{72}(v)$ or $Q_{73}(v)$ which constructs a $Q(v)$ matrix without first computing the eigenvectors of $T(v)$.

The area of the fundamental region of all matrices $Q_{72}(v)$ studied in this paper except for $Q_{72}^{(2)}(v;n\eta)$ with m_{20} odd is determined from the quasiperiodicity relations to be $4KK'$.

However, the fundamental region of $h(v)$ has the area $2KK'$. To see this we first note from (112)–(115) that $h(v)$ has quasi periods ω_1 and ω_2. However it follows further from (118) that for any s_B of the form

$$s_B = \text{even integer } K + \text{integer } iK' \tag{55}$$

that

$$h(v + s_B) = c_1 e^{c_2 v} h(v) \tag{56}$$

where c_1 and c_2 are independent of v. Thus if we write

$$\omega_1/2 = r_1 K + r_2 K' \tag{57}$$

$$\omega_2/2 = bK + aiK' \tag{58}$$

we see that in cases where r_1 is even that $h(v)$ will have quasiperiods $\omega_1/2,\ \omega_2$ and when r_1 is odd that b may be chosen even and thus $h(v)$ will have quasiperiods $\omega_1, \omega_2/2$. The phase factor which is produced on the left hand side of (52) under the quasi periods ω_2 or $\omega_2/2$ is compensated for on the right hand side by the correct choice of the parameter ν which occurs in the exponential factor $e^{i\nu v}$ which is present in the $q_B(v)$ of the factorization (54). Therefore in these cases the area of the fundamental region of $h(v)$ is $2KK'$ which is one half of the area of the fundamental region of $Q_{72}(v)$. If r_1 is even (odd) then s_B is $\omega_1/2$ ($\omega_2/2$).

2.4 Functional Equations for $Q_{72}^{(1)}(v)$ and $\tilde{Q}_{72}^{(2)}(v)$

In [2] and [9] we conjectured and verified in several cases that for m_{10} odd and $m_{20} = 0$ the matrices $Q_{72}^{(1)}(v)$ satisfy the matrix functional equation

$$\begin{aligned}
&\exp\left(-\frac{i\pi N v}{2K}\right) Q_{72}^{(1)}(v - iK') \\
&\quad = A \sum_{l=0}^{L-1} h^N(v - (2l+1)\eta) \frac{Q_{72}^{(1)}(v)}{Q_{72}^{(1)}(v - 2l\eta) Q_{72}^{(1)}(v - 2(l+1)\eta)}
\end{aligned} \tag{59}$$

where A commutes with $Q_{72}^{(1)}(v)$ and is independent of v. We therefore have investigated whether such a functional equation will hold for $Q_{72}^{(1)}(v)$ and $\tilde{Q}_{72}^{(2)}(v)$ for all other values of m_{20}. To make such an investigation we need to generalize the shift in (59) from iK' to

$$s = s_1 K + s_2 iK' \tag{60}$$

and adjust phase factors in (59) to match the quasiperiodicity properties of $Q_{72}^{(1)}(v)$ and $\tilde{Q}_{72}^{(2)}(v)$. Therefore we conjecture for all cases except $\tilde{Q}_{72}^{(2)}(v; n\eta)$ with m_{20} odd that there is a value of s such that the matrix equation holds

$$\begin{aligned}
&\exp\left(-\frac{i\pi(-s_1 r_2 + s_2 r_1) N v}{\omega_1}\right) Q_{72}(v - s) \\
&\quad = A \sum_{l=0}^{L-1} h^N(v - (2l+1)\eta) \frac{Q_{72}(v)}{Q_{72}(v - 2l\eta) Q_{72}(v - 2(l+1)\eta)}
\end{aligned} \tag{61}$$

where $Q_{72}(v)$ is either $Q_{72}^{(1)}(v)$ or $\tilde{Q}_{72}^{(2)}(v)$.

Table 6 Shifts s and s' for $Q_{72}^{(1)}(v)$. In the last column we indicate that discrete symmetry operator which commutes with $Q_{72}^{(1)}(v)$. The two shifts s for m_{10} and m_{20} both odd are equivalent because they differ by a quasi-period

m_{10}	m_{20}	s	s'	Quasiperiods	
Odd	Even	iK'	$2K$	$\omega_1,\ \omega_2$	S
Odd	Odd	$iK', 2K$	$2K+iK'$	$\omega_1/2,\ 2\omega_2$	RS
Even	Odd	$2K$	iK'	$\omega_1/2,\ 2\omega_2$	R

Table 7 Shifts s and s' for $\tilde{Q}^{(2)}(v; n\eta)$ for m_{20} even. In all cases $Q^{(2)}(v; n\eta)$ commutes with S. There is no functional equation for m_{20} odd

m_{10}	m_{20}	s	s'	Quasiperiods
Odd	Even	iK'	$2K$	ω_1, ω_2
Even	Even	iK'	$2K$	ω_1, ω_2

Table 8 Shifts s and s' and quasiperiods for $\tilde{Q}^{(2)}(v; (n+1/2)\eta)$ for m_{10} and m_{20} both even. The entries for m_{10}, m_{20} are $0(2) \equiv$ (mod 4). In the last column we indicate the discrete symmetry operator which commutes with $\tilde{Q}_{72}^{(2)}(v; (n+1/2)\eta)$. The two shifts s for $m_{10} \equiv 0$ and $m_{20} \equiv 2$ (mod 4) are equivalent because they differ by a quasi-period

m_{10}	m_{20}	s	s'	Quasiperiods	
0	0	iK'	$2K$	$\omega_1,\ \omega_2$	S
0	2	$iK', 2K$	$2K+iK'$	$\omega_1,\ \omega_1/2+\omega_2$	RS
2	2	$2K$	iK'	$\omega_1,\ \omega_1/2+\omega_2$	R

We have determined the values of s_1 and s_2 by numerically studying the conjecture in special cases and have found that the functional equation (61) holds for all the matrices $Q_{72}^{(1)}(v)$, $\tilde{Q}_{72}^{(2)}(v; t)$ and $\tilde{Q}^{(2)}(v; (n+1/2)\eta)$ with the single exception of $\tilde{Q}^{(2)}(v; n\eta)$ with m_{20} odd where the matrix shares with $Q_{73}(v)$ the property of commuting with all three symmetry operators S, R and RS. We have also found that there is no shift s for which $Q_{73}(v)$ satisfies the functional equation (61). The values of s_1 and s_2 determined from these studies are given in Tables 6–8.

We also remark that if the shift s is replaced by a shift s' which has the properties

(1) s' is a quasiperiod of $Q_{72}(v)$
(2) the transformation $v \to v + s - s' = v + s_B$ leaves the eigenvalues $q_B^o(v)$ invariant whose $N/2$ roots v_j^B which lie in the fundamental region of $h^N(v)$ are determined by the Bethe's equation (52)

then for the eigenvalues $q_B^o(v)$ the matrix functional equation (61) reduces to scalar functional equations for the eigenvalues $q_B^o(v)$

$$A' \sum_{l=0}^{L-1} h^N(v-(2l+1)\eta) \frac{1}{q_B^o(v-2l\eta)q_B^o(v-2(l+1)\eta)} = 1 \tag{62}$$

with A' is a constant whose value depends on the eigenvalue under consideration. The shifts s' are given in Tables 6–8.

2.5 Comparison of $Q_{72}^{(1)}(v)$ and $\tilde{Q}_{72}^{(2)}(v;t)$ for m_{10} Odd and m_{20} Even

We see from Tables 6 and 7 that when m_{10} is odd and m_{20} is even the matrices $Q_{72}^{(1)}(v)$ and $\tilde{Q}_{72}^{(2)}(v;t)$ (with t chosen as in Table 5) both satisfy the same functional equation (61). Furthermore we saw in Sect. 2.1 that these matrices satisfy the same quasiperiodicity equations (47), (48). It is therefore to be expected that the eigenvectors of these matrices should be the same and that the ratios of the eigenvalues should be independent of v. A numerical study of several special cases reveals that for m_{10} odd and m_{20} even the pair $Q_{72}^{(1)}(v)$ and $\tilde{Q}_{72}^{(2)}(v;t)$ in fact are similar up to proportionality

$$Q_{72}^{(1)}(v) = \text{const}\; M\tilde{Q}_{72}^{(2)}(v;t)M^{-1} \quad m_{10}\text{ odd},\; m_{20}\text{ even} \tag{63}$$

We conjecture that this relation is generally true.

2.6 Comparison of $\tilde{Q}_{72}^{(2)}(v;n\eta)$ and $\tilde{Q}^{(2)}(v;(n+1/2)\eta)$ for $m_{10},\ m_{20} \equiv 0 \pmod 4$

When $m_{10},\ m_{20} \equiv 0 \pmod 4$ we find from Tables 7 and 8 that $\tilde{Q}_{72}^{(2)}(v;n\eta)$ and $\tilde{Q}^{(2)}(v;(n+1/2)\eta)$ also satisfy (61) with the same value of s. Furthermore the quasiperiodicity properties of $\tilde{Q}_{72}^{(2)}(v,n\eta)$ and $\tilde{Q}^{(2)}(v;(n+1/2)\eta)$ are both given by

$$\tilde{Q}_{72}^{(2)}(v+\omega_1) = S^{r_1}\tilde{Q}_{72}^{(2)}(v) \tag{64}$$

$$\tilde{Q}_{72}^{(2)}(v+\omega_2) = S^b q'^{-N} e^{-2\pi i N v/\omega_1}\tilde{Q}_{72}^{(2)}(v) \tag{65}$$

and a numerical study of the case $L_0=3,\ m_{10}=m_{20}=4\ t=0,\ \eta/2,\ N=8$ reveals that $\tilde{Q}_{72}(v;n\eta)$ and $\tilde{Q}^{(2)}(v;(n+1/2)\eta)$ in fact are similar. We conjecture that in general

$$\tilde{Q}_{72}^{(2)}(v;n\eta) = M\tilde{Q}_{72}^{(2)}(v;(n+1/2)\eta)M^{-1} \quad m_{10}, m_{20} \equiv 0 \pmod 4 \tag{66}$$

3 Construction of $Q_{72}^{(2)}(v;t)$ for m_{10} and m_{20} Not Both Even

We treat the steps 1–3 in separate subsections

3.1 The Equation for $TQ_R^{(2)}$ and $Q_L^{(2)}T$

The study of $Q_{72}^{(2)}(v;t)$ for m_{10} and m_{20} not both even closely parallels the study done in [11] and wherever possible we will refer to that paper for details of computations. The principle generalization needed is that for m_{10} and m_{20} both even the dimension L of the local matrices S_R was $L=L_0$ where L_0 is determined from (5) is odd and has no common divisors with m_{10} and m_{20}. In order to treat the cases where m_{10} and m_{20} are not both even we will need to choose L to be an even multiple of L_0 as determined from (5) and to define m_1 and m_2 as (37).

We begin by following [11] to show that when $Q_R^{(2)}(v;t)$ is determined from (16)–(23) that (8) with

$$\omega = \exp\left(\frac{i\pi m_2}{2L}\right) \tag{67}$$

is valid for even L. This is (79) of [11] and in Appendix 3 of [11] it is proven that for all

L, m_1 and m_2 related by (37) that (8) is valid if condition (C.31) of [11] holds

$$(\pm 1)^L \omega^L \frac{\Theta_m[(2L+1)\eta + t]}{\Theta_m(\eta + t)} = 1 \tag{68}$$

For even L (68) becomes

$$(-1)^{m_2/2} = 1 \tag{69}$$

and thus for (8) to hold for even L we have to set $m_2 \equiv 0 \pmod 4$. We thus consider the three cases of Table 1.

The companion matrix $Q_L^{(2)}(v)$ computed from (32) satisfies (9) and is given by (24)–(31).

3.2 The Relation $Q_L(u)\Lambda Q_R(v) = Q_L(v)\Lambda Q_R(u)$

To proceed further we follow [11] and determine sufficient conditions for which the interchange relation (11) holds.

The relation (11) will hold if we can find similarity transformation such that

$$\begin{aligned} &S_L^{(2)}(\alpha,\gamma)_{k,l}(u)\Lambda_{\gamma,\gamma'}S_R^{(2)}(\gamma',\beta)_{k',l'}(v) \\ &\quad = Y_{k,k';m,m'}S_L^{(2)}(\alpha,\gamma)_{m,n}(v)\Lambda_{\gamma,\gamma'}S_R^{(2)}(\gamma',\beta)_{m',n'}(u)Y^{-1}_{n,n';l,l'} \end{aligned} \tag{70}$$

with diagonal matrix Y

$$Y_{k,k';m,m'} = y_{k,k'}\delta_{m,k}\delta_{k',m'} \tag{71}$$

so that

$$\begin{aligned} &S_L^{(2)}(\alpha,\gamma)_{k,l}(u)\Lambda_{\gamma,\gamma'}S_R^{(2)}(\gamma',\beta)_{k',l'}(v) \\ &\quad = \frac{y_{k,k'}}{y_{l,l'}}S_L^{(2)}(\alpha,\gamma)_{k,l}(v)\Lambda_{\gamma,\gamma'}S_R^{(2)}(\gamma',\beta)_{k',l'}(u) \end{aligned} \tag{72}$$

We write

$$S_R^{(2)}(\alpha,\beta)_{m,n} = \Phi^{\alpha}_{m,n}\tau^{\beta}_{m,n}, \qquad S_L^{(2)}(\alpha,\beta)_{m,n} = \tau'^{\alpha}_{m,n}\chi^{\beta}_{m,n} \tag{73}$$

Then

$$S_L(\alpha,\gamma)_{k,l}\Lambda_{\gamma,\gamma'}(u)S_R(\gamma',\beta)_{k'l'}(v) = \tau'^{\alpha}_{k,l}\chi^{\gamma}_{k,l}(u)\Lambda_{\gamma,\gamma'}\Phi^{\gamma'}_{k',l'}(v)\tau^{\beta}_{k',l'} \tag{74}$$

and thus (72) is written for all four cases of Λ as

$$\chi^{\gamma}_{k,l}(u)\Lambda_{\gamma,\gamma'}\Phi^{\gamma'}_{k',l'}(v) = \frac{y_{k,k'}}{y_{l,l'}}\chi^{\gamma}_{k,l}(v)\Lambda_{\gamma,\gamma'}\Phi^{\gamma'}_{k',l'}(u) \tag{75}$$

In Sect. 6 of [11] we explicitly evaluated (75) for the case m_{10} and m_{20} both even. However, this evaluation is also valid as well for all the other cases of m_{10} and m_{20} and we refer the reader to [11] for details of the computation. Thus, in what we hope is a more transparent notation, we find

$$\frac{y_{k,k'}}{y_{k+1,k'+1}} = \frac{\hat{g}(u-v+2t+2(k+k')\eta)}{\hat{g}(v-u+2t+2(k+k')\eta)} \tag{76}$$

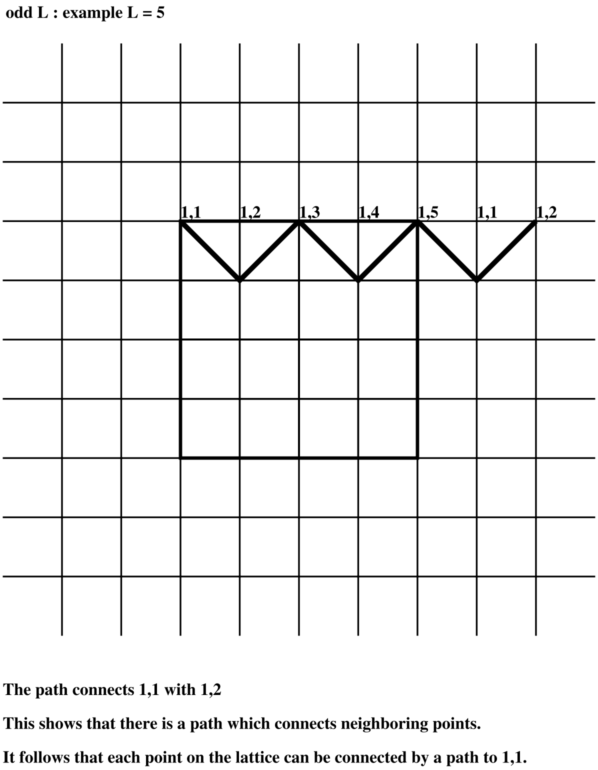
$$\frac{y_{k,k'}}{y_{k+1,k'-1}} = \frac{\tilde{g}(u-v+2(k-k'+1)\eta)}{\tilde{g}(v-u+2(k-k'+1)\eta)} \tag{77}$$

where the definitions of $\hat{g}$, $\tilde{g}$ are given in Table 9 and Appendix 2.

Table 9 Definition of $\hat{g}$ and $\tilde{g}$

A	I	S	R	RS
$\hat{g}$	$g^{-}_{\Theta\Theta}$	$g^{+}_{\Theta\Theta}$	$g^{-}_{\mathrm{H}\Theta}$	$g^{+}_{\mathrm{H}\Theta}$
$\tilde{g}$	$g^{+}_{\Theta\Theta}$	$g^{-}_{\Theta\Theta}$	$g^{+}_{\mathrm{H}\Theta}$	$g^{-}_{\mathrm{H}\Theta}$

Fig. 1 Paths



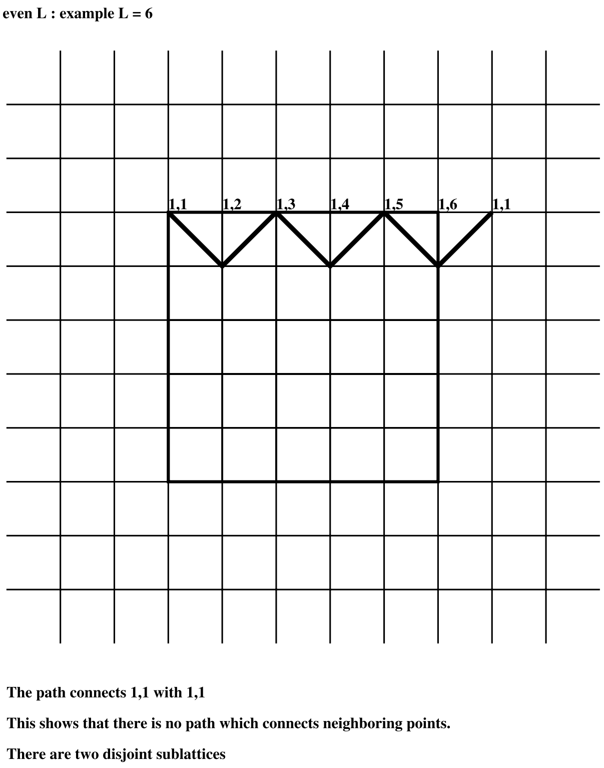
The recursions (76) and (77) are interpreted as describing the transport on a torus of size $L \times L$. Consequently in order to obtain a solution to these equations we must show that from a set of initial values for $y_{k,l}$ all remaining $y_{k,l}$ are determined consistently. This consistency obtains only for certain values of t and the cases of L even and odd must be treated separately.

Consider first the case that L is odd which was treated in [11]. The path shown in Fig. 1 connects an arbitrary point with its neighbor. It follows that all points on the torus can be reached by appropriate paths starting from a single point or that e.g. all $y_{k,l}$ follow from $y_{1,1}$.

If L is even we see from Fig. 2 that there is no path connecting two neighboring points. Equations (76) and (77) thus form two disjoint sets. In this case all $y_{k,l}$ follow from two initial values e.g. $y_{1,1}$ and $y_{1,2}$.

These constructions of all $y_{k,l}$ from one or two initial values will be consistent provided that transport of $y_{k,l}$ on a closed path has the result $y_{k,l}$. There are two cases to consider for L odd and three cases for L even.

Fig. 2 Paths



For any L we find for a closed path on the torus with winding numbers $(1, 1)$

$$y_{k+L,k'+L} = y_{k,k'} \prod_{j=0}^{L-1} \frac{\hat{g}(v-u+2t+2(k+k')\eta+4j\eta)}{\hat{g}(u-v+2t+2(k+k')\eta+4j\eta)} \tag{78}$$

Similarly for any L we find for a closed path for winding numbers $(1, -1)$

$$y_{k+L,k'-L} = y_{k,k'} \prod_{j=0}^{L-1} \frac{\tilde{g}(u-v+2(k-k'+1)\eta+4j\eta)}{\tilde{g}(v-u+2(k-k'+1)\eta+4j\eta)} \tag{79}$$

For even L we have the additional condition that for the path : $(k, k') \to (k+1, k'+1) \to (k, k'+2) \cdots (k, k'+L-2) \to (k+1, k'+L-1) \to (k, k'+L)$ shown in Fig. 2 which has winding numbers $(0, 1)$ we obtain from (76) and (77)

$$y_{k,k'+L} = y_{k,k'} \prod_{j=0}^{L/2-1} \frac{\hat{g}(u-v+2t+2(k+k')\eta+4j\eta)\tilde{g}(u-v+2(k-k'+1)\eta+4j\eta)}{\hat{g}(v-u+2t+2(k+k')\eta+4j\eta)\tilde{g}(v-u+2(k-k'+1)\eta+4j\eta)} \tag{80}$$

Thus for L odd there are two conditions for the existence of a solution of (76) and (77)

$$y_{k+L,k'+L} = y_{k,k'}, \qquad y_{k+L,k'-L} = y_{k,k'} \tag{81}$$

while for L even there are three conditions

$$y_{k,k'+L} = y_{k,k'}, \qquad y_{k+L,k'+L} = y_{k,k'}, \qquad y_{k+L,k'-L} = y_{k,k'} \tag{82}$$

The case of odd L was considered in [11]. Here we consider the case of L even with m_1 and m_2 even as indicated in Table 1. Then using the (anti) symmetry properties (142), (144), (146), (148) we find that (78) and (79) become for all Λ

$$\prod_{j=0}^{L-1} \frac{\hat{g}(v-u+2t+2(k+k')\eta+4j\eta)}{\hat{g}(v-u-2t-2(k+k')\eta-4j\eta)} = 1 \tag{83}$$

$$\prod_{j=0}^{L-1} \frac{\tilde{g}(v-u+2(k-k'+1)\eta+4j\eta)}{\tilde{g}(v-u-2(k-k'+1)\eta-4j\eta)} = 1 \tag{84}$$

and that (80) becomes

$$\prod_{j=0}^{L/2-1} \frac{\hat{g}(u-v+2t+2(k+k')\eta+4j\eta)\tilde{g}(u-v+2(k-k'+1)\eta+4j\eta)}{\hat{g}(u-v-2t-2(k+k')\eta-4j\eta)\tilde{g}(u-v-2(k-k'+1)\eta-4j\eta)} = \pm 1 \tag{85}$$

where for $L/2 =$ even the right hand side is $+1$ for $\Lambda = I, S, R, RS$ and for $L/2 =$ odd the right hand side is $+1$ for $\Lambda = I, S$ and -1 for $\Lambda = R, RS$.

To determine the consistency of (83)–(85) we need the periodicity properties for even m_1 and $m_2 \equiv 0 \pmod 4$ which follow from (141), (143), (145), (147) of Appendix 2

$$g^-_{\Theta\Theta}(u+2L\eta) = g^-_{\Theta\Theta}(u) \qquad g^+_{\Theta\Theta}(u+2L\eta) = g^+_{\Theta\Theta}(u) \tag{86}$$

$$g^-_{\mathrm{H}\Theta}(u+2L\eta) = (-1)^{m_1/2} g^-_{\mathrm{H}\Theta}(u) \qquad g^+_{\mathrm{H}\Theta}(u+2L\eta) = (-1)^{m_1/2} g^+_{\mathrm{H}\Theta}(u) \tag{87}$$

We will treat the cases of $\Lambda = I, S$ and $\Lambda = R, RS$ separately

3.2.1 *The cases* $\Lambda = I, S$

We set

$$t = n\eta \tag{88}$$

and will use the periodicity conditions (86) to prove that in each of (83)–(85) for each factor $\hat{g}$ and $\tilde{g}$ in the numerator there exist values of n such that there is a corresponding factor in the denominator with the same argument modulo the period $4L\eta$ for (83), (84) or $2L\eta$ for (85).

Consider first (83). The difference of the argument of the j factor in the numerator with the j' factor in the denominator is

$$\mathrm{diff}_1(j, j') = 4(n+k+k'+j+j')\eta = \frac{n+k+k'+j+j'}{L} 4L\eta \tag{89}$$

Similarly for (84) the corresponding difference is

$$\mathrm{diff}_2(j, j') = 4(k-k'+1+j+j')\eta = \frac{k-k'+1+j+j'}{L} 4L\eta \tag{90}$$

We see that for fixed integer $0 \le n, k, k', j < L-1$ there is an integer j' with $0 \le j' < L$ such that $\mathrm{diff}_i(j, j')$ with $i = 1, 2$ is an integer multiple of $4L\eta$. This proves that when t is given by (88) with n an integer (83), (84) are satisfied.

We finally consider the more restrictive (85). The difference of arguments of factors $\hat{g}$ are

$$\mathrm{diff}_{h1}(j, j') = 4(n + k + k' + j + j')\eta = \frac{n + k + k' + j + j'}{L/2} 2L\eta \tag{91}$$

and the difference of arguments of functions $\tilde{g}$ is

$$\mathrm{diff}_{h2}(j, j') = 4(k - k' + 1 + j + j')\eta = \frac{k - k' + 1 + j + j'}{L/2} 2L\eta \tag{92}$$

written in a form accommodated to the shorter range $0 \le j' < L/2$. As the period of $\hat{g}$ and $\tilde{g}$ is $2L\eta$ as shown in (86) we see that when n is an integer that (85) is satisfied. We conclude that when t is given by (88) with n integer the interchange relation (11) for $\Lambda = I$ and $\Lambda = S$ is satisfied in all cases listed in Table 1. It thus follows from (14) that $Q^{(2)}(v)$ commutes with S.

3.2.2 *The cases* $\Lambda = R, RS$

The interchange relation (11) for cases $\Lambda = R, RS$ is examined using the (anti)periodicity conditions (87). The proof of (83) and (84) given above for $\Lambda = I, S$ required the periodicity of $4L\eta$ and thus the identical proof works for $\Lambda = R, RS$. However the proof of (85) depends on whether $m_1/2$ and $L/2$ are even or odd and these cases will be treated separately.

Cases II and III of Table 1

We see from Table 1 that $m_1/2$ is even for the cases II and III and thus it follows from (87) that $g^{\pm}_{\mathrm{H}\Theta}(u)$ have the period $2L\eta$. We also see from Table 1 in cases II and III that $L/2$ is even and thus the right hand side of (85) is $+1$. Therefore the proof given in Sect. 3.2.1 works also in this case. We conclude that for cases II and III the interchange relation (11) holds for $A = R$ when t is given by (88) with n integer. Thus from (14) $Q^{(2)}(v; n\eta)$ commutes with R as well as with S (and hence also with RS)

Case I of Table 1 with $L/2 = L_0$ even

We see from Table 1 that $m_1/2 = m_{10}$ is odd for case I and thus it follows from (87) that $g^{\pm}_{\mathrm{H}\Theta}(u)$ are antiperiodic with the period $2L\eta$. When $L/2 = L_0$ is even the right hand side of (85) is $+1$.

To find values of n such that (85) will hold in this case consider $\hat{g}(u - v + 2n\eta + 2(k + k')\eta + 4j\eta)$ in the numerator and $\hat{g}(u - v - 2n\eta - 2(k + k')\eta - 4j'\eta)$ in the denominator. The difference of their arguments is

$$\mathrm{diff}_1(j, j') = 4(n + k + k' + j + j')\eta = \frac{n + k + k' + j + j'}{L/2} 2L\eta \tag{93}$$

As $L/2$ is an integer then for any integer n the difference $\mathrm{diff}_1(j, j')$ may become $\hat{m}2L\eta$, where $\hat{m}$ is an integer. If $\hat{m} =$ even the respective functions drop out of the product on the left hand side of (85). If $\hat{m}$ = odd the two functions drop out up to a minus sign. If for a pair j, j' with $j' \neq j$ the functions drop out there is another pair of functions j', j which also drop out. It follows that if $j' \neq j$ two pairs will drop out without sign change. So we have only to inspect the case $j' = j$.

Similarly consider a function $\tilde{g}(u - v + 2(k - k' + 1)\eta + 4j\eta)$ in the numerator and a function $\tilde{g}(u - v - 2(k - k' + 1)\eta - 4j'\eta)$ in the denominator. The difference of their arguments is

$$\text{diff}_2(j, j') = 4(k - k' + 1 + j + j')\eta = \frac{k - k' + 1 + j + j'}{L/2} 2L\eta \tag{94}$$

Thus for integer n the difference $\text{diff}_2(j, j')$ may become $\tilde{m}2L\eta$, where $\tilde{m}$ is an integer. If $\tilde{m} =$ even the respective functions drop out of the product on the left hand side of (85). If $\tilde{m} =$ odd the two functions drop out up to a minus sign. Thus as before it follows that if $j' \neq j$ two pairs will drop out without sign change. So we have only to inspect the case $j' = j$.

When $j = j'$ and n even then because $L/2$ is even we see that if $k + k'$ is even (odd) there are two (zero) solutions j with $0 \leq j < L/2$ of

$$\frac{n + k + k' + 2j}{L/2} = \text{integer} \tag{95}$$

and if there are two solutions one of these produces a factor -1 and the other a factor $+1$. Similarly if $k + k'$ (and thus $k - k'$ is even (odd)) there are zero (two) solutions j of

$$\frac{k - k' + 1 + 2j}{L/2} = \text{integer} \tag{96}$$

and if there are two solutions these produce a factor -1 and the other a factor $+1$. This results in a factor -1 after all functions on the left hand side of (85) have dropped out and it follows that for $m_1/2 = \text{odd}$ and $L/2 = \text{even}$ the interchange relation is not satisfied for $A = R$ and $A = RS$ if $t = 2l\eta$.

In the opposite case $j = j'$ and n odd Then for $k + k'$ even (odd) there are zero (two) or two solutions j with $0 \leq j < L/2$ of

$$\frac{n + k + k' + 2j}{L/2} = \text{integer} \tag{97}$$

and zero (two) or two solutions j of

$$\frac{k - k' + 1 + 2j}{L/2} = \text{integer} \tag{98}$$

Thus all factors -1 appear in pairs. It follows that for $m_1/2 = \text{odd}$ and $L/2 = \text{even}$ the interchange relation is satisfied for $A = R$ and $A = RS$ if $t = (2l + 1)\eta$.

Case I of Table 1 with $L/2 = L_0$ odd

In this case the right hand side of (85) is -1. This is the only difference with the case $m_1/2$ odd and $L/2$ even. It follows that the results of Sect. 3.2.2 are reversed. Thus for $t = (2l + 1)\eta$ the interchange relation is valid for $A = I, S$ and for $t = 2l\eta$ the interchange relation is valid for $A = I, S, R, RS$.

3.3 The Matrix $Q^{(2)}(v; n\eta)$

It remains to compute $Q^{(2)}(v;t)$ from $Q_R^{(2)}(v)$ by using

$$Q^{(2)}(v;t) = Q_R^{(2)}(v) Q_R^{(2)-1}(v_0) \tag{99}$$

and for this construction to be valid the matrix $Q_R^{(2)}(v)$ must be non singular for some value of v. While no analytic results are available we have investigated this question numerically for examples of all three cases of Table 1 for systems of size $N = 8$. The conclusions of this study are given in Table 5.

From the validity of the interchange relation for all Λ it follows for m_{20} odd that

$$[Q^{(2)}(v), S] = [Q^{(2)}(v), R] = [Q^{(2)}(v), RS] = 0 \tag{100}$$

which are the same symmetry properties of the transfer matrix $T(v)$.

For m_{10} odd and m_{20} even and the choice of t given in Table 5 the commutation properties $Q^{(2)}(v;t)$ are

$$[Q^{(2)}(v;t), S] = 0, \qquad [Q^{(2)}(v;t), R] \neq 0, \qquad [Q^{(2)}(v;t), RS] \neq 0 \tag{101}$$

4 Discussion and Open Questions

The studies of Q matrices, beginning with [1, 7] and continuing through [2, 8–11] have revealed that the concept of a Q matrix is not unique and that for a full understanding it is necessary to study several essentially different constructions. For example the construction 1973 [7] exists for generic η for an even number of sites and commutes with both symmetry operators S and R but is of limited use in determining the degeneracy of the transfer matrix eigenvalues at roots of unity. The Q matrix defined in the 1972 paper [1] is defined for all N and, because it fails to commute with the operator R, is very useful in characterizing the degeneracies of the transfer matrix but does not exist when m_{10} and m_{20} are both even [11]. For this excluded case new Q matrices, which exist only for N even, were found in [10, 11] and [21] and these new Q matrices are shown in [10] and [11] to reveal the full degeneracy of the transfer matrix eigenvalues. It must be emphasized, however, that because all of these Q matrices satisfy the same matrix TQ equation all constructions of Q lead to the same values for the eigenvalues of $T(v)$. In particular the free energy as calculated in [1] is not affected by any of the considerations of [2, 8–11].

Particularly for the most important case of real η it seems somewhat misleading and unnatural that different forms of Q should be used for different classes of roots of unity. In this paper we have overcome this dichotomy for even N by demonstrating that the new Q matrix of [10] and [11] exists for all roots of unity. Furthermore we have extended the functional equation for $Q_{72}(v)$ first conjectured in [2] for the special case $m_{20} = 0$ and m_{10} odd to all values of m_{10} and m_{20} where a matrix $Q_{72}(v)$ constructed by the method of [2] exists.

However, there are several open questions which have been raised that need further investigation:

(1) In both [11] and the present paper we have found the commutation of the Q matrices with the discrete symmetry operators depends on the parity of the numbers m_{10} and m_{20}. This property has not been anticipated in the literature and needs further explanation.

(2) The discovery in this paper that for some cases Q matrices constructed by the method of [1] can have degenerate eigenvalues is totally unexpected.
(3) For the case $m_{20}=0$ and m_{10} even it was observed in [21] that the $Q_{72}(v)$ matrix is intimately connected with the construction used by Baxter in [5–7] to obtain the SOS models and the eigenvectors of the eight vertex model. Furthermore in [21] this connection is exploited to prove the functional equation for Q. It is thus most interesting to see whether this connection with [5–7] extends to all the matrices $Q_{72}^{(2)}(v;t)$ considered in this paper. This is particularly interesting in the case $m_{20}=0$ and m_{10} odd where $\tilde{Q}_{72}^{(2)}(v;n\eta)$ is similar up to proportionality to $Q_{72}^{(1)}(v)$ because $Q_{72}^{(1)}(v)$ is defined for all N and not just N even.
(4) The matrix of 1973 [7] which exists for generic η and the new Q matrix of [10, 11] and [21] which is shown in this present paper to exist for all roots of unity are only defined for N even. The matrix of 1972 [1] exists for all N but only when m_{10} and m_{20} are not both even. Consequently there is as yet NO Q matrix for the case of m_{10} and m_{20} both even and N odd. This is perhaps the most interesting and challenging of all the cases of the eight vertex model [9, 16–20].
(5) Finally we wish to draw attention to an important difference between the scalar tq equations of [3–7, 15] and the matrix TQ equation of [1, 2, 8–11] and this present paper. For the scalar tq equation it is emphasized in [3] and [15] that the centers of the complete L strings are completely arbitrary because these strings cancel out of the scalar tq equation. This arbitrariness and cancellation is what is referred to in [22] as the "incompleteness of Bethe's equation". Thus the solutions of the scalar tq equation can have as many arbitrary parameters as there are complete L strings. This is to be contrasted with all known $Q(v)$ matrices for the 8 vertex model which satisfy the matrix TQ equation (and the commutation relation (4)). None of these $Q(v)$ matrices contains even one arbitrary continuous parameter which affects only the eigenvalues of $Q(v)$ but not the eigenvalues of $T(v)$. We do note, however, that in [10] it is shown that for $m_2=0$ the matrix $Q_{72}^{(2)}(v;t)$ for all t satisfies (1) and the commutation relation (3) but fails to satisfy (4). It is an open question whether a $Q(v)$ matrix with such arbitrary continuous parameters can be found which satisfies all three of the conditions (1), (3) and (4) without having first computed the eigenvectors of $T(v)$.

Appendix 1: The Functions $\mathsf{H}_m(v)$, $\Theta_m(v)$ and the Transfer Matrix $T(v)$

The standard definition of the theta functions $\mathsf{H}(v)$ and $\Theta(v)$ are

$$\mathsf{H}(v)=2\sum_{n=1}^{\infty}(-1)^{n-1}q^{(n-\frac{1}{2})^2}\sin[(2n-1)\pi v/(2K)] \tag{102}$$

$$\Theta(v)=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(nv\pi/K) \tag{103}$$

where

$$q=e^{-\pi K'/K} \tag{104}$$

In this paper we will use the "modified" theta functions $\mathsf{H}_m(v)$ and $\Theta_m(v)$ of [7] defined by

$$\mathsf{H}_m(u)=\exp\left(\frac{i\pi m_{20}(u-K)^2}{8KL_0\eta}\right)\mathsf{H}(u), \qquad \Theta_m(u)=\exp\left(\frac{i\pi m_{20}(u-K)^2}{8KL_0\eta}\right)\Theta(u) \tag{105}$$

The functions $\mathsf{H}_m(v)$ and $\Theta_m(v)$ are themselves theta functions [11] with (quasi)periods $\omega_{1,2}$ given in (28)–(31) [11] by

$$\mathsf{H}_m(v+\omega_1)=(-1)^{r_1+r_1r_2}\mathsf{H}_m(v) \tag{106}$$

$$\Theta_m(v+\omega_1)=(-1)^{r_1r_2}\Theta_m(v) \tag{107}$$

$$\begin{aligned}\mathsf{H}_m(v+\omega_2)&=(-1)^{b+ab}q'^{-1}\exp\left(-\frac{2\pi i(v-K)}{\omega_1}\right)\mathsf{H}_m(v)\\&=(-1)^{a+b+ab}q'^{-(1+r_2)}\exp\left(-\frac{2\pi iv}{\omega_1}\right)\mathsf{H}_m(v)\end{aligned} \tag{108}$$

$$\begin{aligned}\Theta_m(v+\omega_2)&=(-1)^{ab}q'^{-1}\exp\left(-\frac{2\pi i(v-K)}{\omega_1}\right)\Theta_m(v)\\&=(-1)^{a+ab}q'^{-(1+r_2)}\exp\left(-\frac{2\pi iv}{\omega_1}\right)\Theta_m(v)\end{aligned} \tag{109}$$

We can slightly simplify these relations if we note firstly from the definition (40) that a and b cannot both be even and thus $a+b+ab$ must be odd. Thus

$$(-1)^{a+b+ab}=-1 \quad \text{and} \quad (-1)^{a+ab}=(-1)^{1+b} \tag{110}$$

Furthermore, since by definition $(r_1,r_2)=1$ the identical argument shows that

$$(-1)^{r_1+r_2+r_1r_2}=-1 \quad \text{and} \quad (-1)^{r_1+r_1r_2}=(-1)^{1+r_2} \tag{111}$$

Thus we will use (106)–(109) in the slightly simpler form

$$\mathsf{H}_m(v+\omega_1)=(-1)^{1+r_2}\mathsf{H}_m(v) \tag{112}$$

$$\Theta_m(v+\omega_1)=(-1)^{r_1r_2}\Theta_m(v) \tag{113}$$

$$\mathsf{H}_m(v+\omega_2)=-q'^{-(1+r_2)}\exp\left(-\frac{2\pi iv}{\omega_1}\right)\mathsf{H}_m(v) \tag{114}$$

$$\Theta_m(v+\omega_2)=(-1)^{1+b}q'^{-(1+r_2)}\exp\left(-\frac{2\pi iv}{\omega_1}\right)\Theta_m(v) \tag{115}$$

We also recall (109)–(114) of [11]

$$H_m(2K-v)=H_m(v), \qquad \Theta_m(2K-v)=\Theta_m(v) \tag{116}$$

$$\begin{aligned}H_m(-v)&=-\exp\left(\frac{i\pi m_2 v}{2L\eta}\right)H_m(v),\\\Theta_m(-v)&=\exp\left(\frac{i\pi m_2 v}{2L\eta}\right)\Theta_m(v)\end{aligned} \tag{117}$$

$$\begin{aligned}\Theta_m(v+iK')&=iq^{-1/4}\exp\left(\frac{-i\pi m_1 v}{2L\eta}\right)CH_m(v)\\H_m(v+iK')&=iq^{-1/4}\exp\left(\frac{-i\pi m_1 v}{2L\eta}\right)C\Theta_m(v)\end{aligned} \tag{118}$$

where

$$C = \exp\left(\frac{\pi m_2 K'}{8KL\eta}(2K - iK')\right) \tag{119}$$

$$H_m(u + 2L_0\eta) = (-1)^{m_{10}} i^{m_{10}m_{20}} \begin{cases} H_m(u) & \text{if } m_{20} = \text{even} \\ \Theta_m(u) & \text{if } m_{20} = \text{odd} \end{cases} \tag{120}$$

$$\Theta_m(u + 2L_0\eta) = i^{m_{10}m_{20}} \begin{cases} \Theta_m(u) & \text{if } m_{20} = \text{even} \\ \mathsf{H}_m(u) & \text{if } m_{20} = \text{odd} \end{cases} \tag{121}$$

For odd m_{20} the integers r_0 and r_2 are odd and r_1 is even. Thus

$$2L_0\eta = \omega_1/2 + \omega_1(r_0 - 1)/2 \tag{122}$$

with $(r_0 - 1)/2$ integer and using (39), (112) and (113) we write (120) and (121) as

$$\begin{aligned} \mathsf{H}_m\left(v + \frac{\omega_1}{2}\right) &= (-1)^{r_1/2} \exp\left(\frac{i\pi r_1 r_2}{4}\right)\Theta_m(v) \\ \Theta_m\left(v + \frac{\omega_1}{2}\right) &= \exp\left(\frac{i\pi r_1 r_2}{4}\right)\mathsf{H}_m(v) \end{aligned} \tag{123}$$

The modified theta functions have the following identities:

$$\begin{aligned} &\Theta_m(u)\Theta_m(v) + \mathsf{H}_m(u)\mathsf{H}_m(v) \\ &\quad = \frac{2q^{1/4}}{\mathsf{H}(K)\Theta(K)} \exp\left(\frac{i\pi m_{20} K'^2}{8KL\eta}\right) \\ &\quad \times \mathsf{H}_m((u + v + iK')/2)\mathsf{H}_m((u + v - iK')/2) \\ &\quad \times \mathsf{H}_m((iK' + u - v)/2 + K)\mathsf{H}_m((iK' - u + v)/2 + K) \end{aligned} \tag{124}$$

$$\begin{aligned} &\Theta_m(u)\Theta_m(v) - \mathsf{H}_m(u)\mathsf{H}_m(v) \\ &\quad = \frac{2q^{1/4}}{\mathsf{H}(K)\Theta(K)} \exp\left(\frac{i\pi m_{20}(K'^2 - 4K^2)}{8KL\eta}\right) \\ &\quad \times \mathsf{H}_m((iK' + u - v)/2)\mathsf{H}_m((iK' - u + v)/2) \\ &\quad \times \mathsf{H}_m((u + v + iK')/2 + K)\mathsf{H}_m((u + v - iK')/2 - K) \end{aligned} \tag{125}$$

$$\begin{aligned} &\Theta_m(u)\mathsf{H}_m(v) + \mathsf{H}_m(u)\Theta_m(v) \\ &\quad = \frac{2}{\mathsf{H}(K)\Theta(K)} \\ &\quad \times \mathsf{H}_m((u + v)/2)\Theta_m((u + v)/2)\mathsf{H}_m((u - v)/2 + K)\Theta_m((u - v)/2 + K) \end{aligned} \tag{126}$$

$$\begin{aligned} &\mathsf{H}_m(u)\Theta_m(v) - \Theta_m(u)\mathsf{H}_m(v) \\ &\quad = \frac{2}{\mathsf{H}(K)\Theta(K)} \exp\left(\frac{-i\pi m_{20} K}{2L\eta}\right) \\ &\quad \times \mathsf{H}_m((u - v)/2)\Theta_m(-(u - v)/2) \\ &\quad \times \mathsf{H}_m((u + v)/2 + K)\Theta_m((u + v)/2 - K) \end{aligned} \tag{127}$$

The transfer matrix of the eight vertex model is [7]

$$T(v)|_{\alpha,\beta} = \mathrm{Tr} W(\alpha_1, \beta_1) W(\alpha_2, \beta_2) \cdots W(\alpha_N, \beta_N) \tag{128}$$

where the Boltzmann weights $W(\alpha, \beta)$ are 2×2 matrices with the non zero matrix elements given by

$$W(1,1)|_{1,1} = W(-1,-1)|_{-1,-1} = \Theta_m(-2\eta)\Theta_m(\eta - v)\mathsf{H}_m(\eta + v) \tag{129}$$
$$W(-1,-1)|_{1,1} = W(1,1)|_{-1,-1} = -\Theta_m(-2\eta)\mathsf{H}_m(\eta - v)\Theta_m(\eta + v) \tag{130}$$
$$W(-1,1)|_{1,-1} = W(1,-1)|_{-1,1} = -\mathsf{H}_m(-2\eta)\Theta_m(\eta - v)\Theta_m(\eta + v) \tag{131}$$
$$W(1,-1)|_{1,-1} = W(-1,1)|_{-1,1} = \mathsf{H}_m(-2\eta)\mathsf{H}_m(\eta - v)\mathsf{H}_m(\eta + v) \tag{132}$$

Appendix 2: The functions $f^{\pm}_{\Theta\Theta}(u)$, $f^{\pm}_{\mathsf{H}\Theta}(u)$, $g^{\pm}_{\Theta\Theta}(u)$, $g^{\pm}_{\mathsf{H}\Theta}(u)$

We recall the definitions of [11]

$$\begin{aligned} f^{+}_{\Theta\Theta}(u) &= \mathsf{H}_m((u + iK')/2)\mathsf{H}_m((u - iK')/2) \\ g^{+}_{\Theta\Theta}(u) &= \mathsf{H}_m((iK' + u)/2 + K)\mathsf{H}_m((iK' - u)/2 + K) \end{aligned} \tag{133}$$

$$\begin{aligned} g^{-}_{\Theta\Theta}(u) &= -\mathsf{H}_m((iK' + u)/2)\mathsf{H}_m((iK' - u)/2) \\ f^{-}_{\Theta\Theta}(u) &= -\mathsf{H}_m((iK' + u)/2 + K)\mathsf{H}_m((u - iK')/2 - K) \end{aligned} \tag{134}$$

$$\begin{aligned} f^{+}_{\mathsf{H}\Theta}(u) &= \mathsf{H}_m(u/2)\Theta_m(u/2) \\ g^{+}_{\mathsf{H}\Theta}(u) &= \mathsf{H}_m(u/2 + K)\Theta_m(u/2 + K) \end{aligned} \tag{135}$$

$$\begin{aligned} g^{-}_{\mathsf{H}\Theta}(u) &= \mathsf{H}_m(u/2)\Theta_m(-u/2) \\ f^{-}_{\mathsf{H}\Theta}(u) &= \mathsf{H}_m(u/2 + K)\Theta_m(u/2 - K) \end{aligned} \tag{136}$$

$$\begin{aligned} \Theta_m(u)\Theta_m(v) + \mathsf{H}_m(u)\mathsf{H}_m(v) &= \frac{2q^{1/4}}{\mathsf{H}_1(0)\Theta_1(0)} \exp\left(\frac{i\pi m_2 K'^2}{8KL\eta}\right) \\ &\quad \times f^{+}_{\Theta\Theta}(u + v) g^{+}_{\Theta\Theta}(u - v) \end{aligned} \tag{137}$$

$$\begin{aligned} \Theta_m(u)\Theta_m(v) - \mathsf{H}_m(u)\mathsf{H}_m(v) &= \frac{2q^{1/4}}{\mathsf{H}_1(0)\Theta_1(0)} \exp\left(\frac{i\pi m_2 (K'^2 - 4K^2)}{8KL\eta}\right) \\ &\quad \times g^{-}_{\Theta\Theta}(u - v) f^{-}_{\Theta\Theta}(u + v) \end{aligned} \tag{138}$$

$$\Theta_m(u)\mathsf{H}_m(v) + \mathsf{H}_m(u)\Theta_m(v) = \frac{2}{\mathsf{H}_1(0)\Theta_1(0)} f^{+}_{\mathsf{H}\Theta}(u + v) g^{+}_{\mathsf{H}\Theta}(u - v) \tag{139}$$

$$\begin{aligned} \mathsf{H}_m(u)\Theta_m(v) - \Theta_m(u)\mathsf{H}_m(v) &= \frac{2}{\mathsf{H}_1(0)\Theta_1(0)} \exp\left(\frac{-i\pi m_2 K}{2L\eta}\right) \\ &\quad \times g^{-}_{\mathsf{H}\Theta}(u - v) f^{-}_{\mathsf{H}\Theta}(u + v) \end{aligned} \tag{140}$$

For m_1 and m_2 both even it is proven in [11] that

$$g^{-}_{\Theta\Theta}(u + 2L\eta) = (-1)^{m_2/2}(-1)^{m_1 m_2/4} g^{-}_{\Theta\Theta}(u) \tag{141}$$

$$g^{-}_{\Theta\Theta}(-u) = g^{-}_{\Theta\Theta}(u) \tag{142}$$

$$g^{+}_{\Theta\Theta}(u+2L\eta) = (-1)^{m_1 m_2/4} g^{+}_{\Theta\Theta}(u) \tag{143}$$

$$g^{+}_{\Theta\Theta}(-u) = g^{+}_{\Theta\Theta}(u) \tag{144}$$

$$g^{-}_{\mathrm{H}\Theta}(u+2L\eta) = (-1)^{(m_1+m_2)/2}(-1)^{m_1 m_2/4} g^{-}_{\mathrm{H}\Theta}(u) \tag{145}$$

$$g^{-}_{\mathrm{H}\Theta}(-u) = -g^{-}_{\mathrm{H}\Theta}(u) \tag{146}$$

$$g^{+}_{\mathrm{H}\Theta}(u+2L\eta) = (-1)^{m_1/2}(-1)^{m_1 m_2/4} g^{+}_{\mathrm{H}\Theta}(u) \tag{147}$$

$$g^{+}_{\mathrm{H}\Theta}(-u) = g^{+}_{\mathrm{H}\Theta}(u) \tag{148}$$

Appendix 3: Quasiperiodicity

We here derive the quasiperiodicity relations (44), (45), (49), (50) for $Q^{(2)}(v;n\eta)$ for m_{10} and m_{20} not both even and review several of the computations in [11].

$Q^{(2)}_{72}(v;n\eta)$ for m_{10} odd and m_{20} even

The quasiperiodicity properties of $Q^{(2)}_{72}(v;n\eta)$ for m_{10} odd are derived in an identical fashion to the case with m_{10} even in [11]. In both cases r_0 is even and we find from (246) and (250) of [11] (and the last line of (108) and (109)) that

$$S^{(2)}_R(\alpha,\beta)(v+\omega_1) = (-\alpha)^{r_1}(-1)^{r_1 r_2} S^{(2)}_R(\alpha,\beta)(v) \tag{149}$$

and

$$S^{(2)}_R(\alpha,\beta)(v+\omega_2) = (-1)^{nr_0/2}\alpha^b(-1)^{a+b+ab} q'^{-(1+r_2)} e^{-2\pi i v/\omega_1} M S_R(\alpha,\beta)(v) M^{-1} \tag{150}$$

with

$$M_{k,k'} = \delta_{k,k'} e^{-\pi i r_0 k(k-1)/(2L)}(-1)^{nr_0 k/2} e^{-\pi i n r_0 k/(2L)} \tag{151}$$

Thus using (6) and (10) (and the fact that N is even) we find for m_{10} either even or odd

$$Q^{(2)}_{72}(v+\omega_1;n\eta) = S^{r_1} Q^{(2)}_{72}(v;n\eta) \tag{152}$$

$$Q^{(2)}_{72}(v+\omega_2:n\eta) = q'^{-N(1+r_2)} e^{-2\pi i N v/\omega_1} S^b Q(v;n\eta) \tag{153}$$

In the case where m_{10} is odd it follows from (39) that r_1 is odd and (152) reduces to (44).

$Q^{(2)}_{72}(v;n\eta)$ for m_{20} odd

In this case we recall that r_0 and r_2 are odd and r_1 is even.

Proof of relation (49)

For convenience we set

$$C = \exp\left(\frac{i\pi r_1 r_2}{4}\right) \tag{154}$$

and find from (123) that

$$S^{(2)}_R(\alpha,\beta)_{k,k+1}\left(v+\frac{\omega_1}{2}\right) = iC(\sigma_1\sigma_3^{r_1/2})_{\alpha,\gamma}(iS^{(2)}_R(\gamma,\beta)_{k,k+1}(v)) \tag{155}$$

$$S_R^{(2)}(\alpha,\beta)_{k+1,k}\left(v+\frac{\omega_1}{2}\right)=iC(\sigma_1\sigma_3^{r_1/2})_{\alpha,\gamma}(-iS_R^{(2)}(\gamma,\beta)_{k,k+1}(v)) \tag{156}$$

We perform a similarity transformation to remove the factors $\pm i$ in front of $S_R^{(2)}$ on the right hand side.

$$\tilde{S_R^{(2)}}(\alpha,\beta)_{k,l}(v)=A_{kk'}S_R^{(2)}(\alpha,\beta)_{k'l'}(v)A_{l',l}^{-1} \tag{157}$$

where A is a diagonal $L\times L$ matrix.

$$A_{kl}=a_k\delta_{kl} \tag{158}$$

$$\tilde{S}_R^{(2)}(\alpha,\beta)_{k,k+1}\left(v+\frac{\omega_1}{2}\right)=iC(\sigma_1\sigma_3^{r_1})_{\alpha,\gamma}\left(i\frac{a_k}{a_{k+1}}S_R^{(2)}(\gamma,\beta)_{k,k+1}(v)\right) \tag{159}$$

$$\tilde{S_R}^{(2)}(\alpha,\beta)_{L,1}\left(v+\frac{\omega_1}{2}\right)=iC(\sigma_1\sigma_3^{r_1})_{\alpha,\gamma}\left(i\frac{a_L}{a_1}S_R^{(2)}(\gamma,\beta)_{k,k+1}(v)\right) \tag{160}$$

In (159) we set

$$a_{k+1}=ia_k \tag{161}$$

from which it follows that

$$a_k=i^{k-1}a_1 \tag{162}$$

and thus in (160)

$$i\frac{a_L}{a_1}=i^L=1 \tag{163}$$

because $L=4L_0$. Thus we find

$$\tilde{S}_R^{(2)}(\alpha,\beta)_{k,l}\left(v+\frac{\omega_1}{2}\right)=iC(\sigma_1\sigma_3^{r_1/2})_{\alpha,\gamma}S_R^{(2)}(\gamma,\beta)_{k,l}(v) \tag{164}$$

which gives (49) when inserted into (10).

Proof of relation (50)

We find from (114) and (115) that with $C(v)=\exp(-2\pi iv/\omega_1)$

$$\begin{aligned}S_R(\alpha,\beta)_{k,k+1}(v+\omega_2)={}&-q'^{-(1+r_2)}C(v)\exp\left(\frac{4\pi ik\eta}{\omega_1}\right)\exp\left(\frac{2\pi it}{\omega_1}\right)\\&\times(\sigma_3)^b_{\alpha,\gamma}S_R(\gamma,\beta)_{k,k+1}(v)\end{aligned} \tag{165}$$

$$\begin{aligned}S_R(\alpha,\beta)_{k+1,k}(v+\omega_2)={}&-q'^{-(1+r_2)}C(v)\exp\left(-\frac{4\pi ik\eta}{\omega_1}\right)\exp\left(-\frac{2\pi it}{\omega_1}\right)\\&\times(\sigma_3)^b_{\alpha,\gamma}S_R(\gamma,\beta)_{k+1,k}(v)\end{aligned} \tag{166}$$

We perform a similarity transformation to remove the expressions $\exp(\pm 4\pi ik\eta/\omega_1)$ $\exp(\pm\frac{2\pi it}{\omega_1})$ on the right hand sides.

$$\tilde{S}_R(\alpha,\beta)_{k,l}(v)=A_{kk'}S_R(\alpha,\beta)_{k'l'}(v)A_{l',l}^{-1} \tag{167}$$

where A is a diagonal $L \times L$ matrix.

$$A_{kl} = a_k \delta_{kl} \tag{168}$$

$$a_{k+1} = \exp\left(\frac{4\pi i k\eta}{\omega_1}\right) \exp\left(\frac{2\pi i t}{\omega_1}\right) a_k \tag{169}$$

It follows that

$$a_k = \exp\left(\frac{2\pi i \eta k(k-1)}{\omega_1}\right) \exp\left(\frac{2\pi i t(k-1)}{\omega_1}\right) a_1 \tag{170}$$

Which removes the expressions $\exp(\pm 4\pi i k\eta/\omega_1)\exp(\pm 2\pi i t/\omega_1)$ in (165) and (166) for $k < L$. For $k = L$ we need that

$$\frac{a_L}{a_1} \exp(4\pi i L\eta/\omega_1)\exp(2\pi i t\eta/\omega_1) = 1 \tag{171}$$

must be satisfied. We set $t = n\eta$. Then

$$\frac{a_L}{a_1} \exp(4\pi i L\eta/\omega_1)\exp(2\pi i n\eta/\omega_1) = \exp(2\pi i r_0 L(L+1)/(4L_0))\exp(2\pi i n r_0 L/(4L_0)) \tag{172}$$

Thus, noting from Table 3 for m_{20} odd that $L = 4L_0$ we see that (171) holds. Therefore we have shown that

$$\tilde{S}_R(\alpha,\beta)_{k,l}(v+\omega_2) = -q'^{-(1+r_2)} \exp\left(-\frac{2\pi i v}{\omega_1}\right) \sigma^b_{3\,\alpha,\gamma} S_R(\gamma,\beta)_{k,l}(v) \tag{173}$$

which gives (50) when used in (10).

Quasiperiodicity for $Q^{(1)}_{72}(v)$ for m_{10} odd and m_{20} even

It follows from (39) when m_{10} is odd and m_{20} is even that r_0 is even and r_1 is odd. Therefore (271) of [11] becomes

$$S^{(1)}_R(\alpha,\beta)(v+\omega_1) = (-1)^{1+r_2}\sigma_{3\alpha,\gamma} S^{(1)}_R(\gamma,\beta) \tag{174}$$

and using (6) and (10) we obtain

$$Q^{(1)}_{72oe}(v+\omega_1) = (-1)^{N(1+r_2)} S Q^{(1)}_{72oe}(v) \tag{175}$$

From (220) of [11] and (110) we have

$$S^{(1)}_R(\alpha,\beta)(v+\omega_2) = (-1)^{(1+a)} q'^{-1} \exp\left(-\frac{2\pi i v}{\omega_1}\right) M^{(1)} \sigma^b_{3\alpha,\gamma} S^{(1)}_R(\gamma,\beta)(v) M^{(1)-1} \tag{176}$$

with

$$M^{(1)}_{k,k'} = e^{-2\pi i \eta k(k-1)/\omega_1} \delta_{k,k'} \tag{177}$$

Inserting (176) into (10) and using (6) we obtain

$$Q^{(1)}_{72}(v+\omega_2) = (-1)^{N(1+a)} q'^{-N} \exp\left(-\frac{2\pi i N v}{\omega_1}\right) S^b Q^{(1)}_{72}(v) \tag{178}$$

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